ASYMPTOTIC BEHAVIOURS OF SOLUTIONS FOR FINITE DIFFERENCE ANALOGUE OF THE CHIPOT-WEISSLER EQUATION

by

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Abstract. — This paper deals with nonlinear parabolic equation for which a local solution in time exists and then blows up in a finite time. We consider the Chipot-Weissler equation:

$$u_t = u_{xx} + u^p - |u_x|^q\,, \quad x \in (-1,1);\ t > 0, \quad p > 1 \text{ and } 1 \leq q \leq \frac{2p}{p+1}.$$

We study the numerical approximation, we show that the numerical solution converges to the continuous one under some restriction on the initial data and the parameters p and q. Moreover, we study the numerical blow up sets and we show that although the convergence of the numerical solution is guaranteed, the numerical blow up sets are sometimes different from that of the PDE.

Contents

1. Introduction

In this paper, we consider the nonlinear parabolic problem

$$\begin{cases} u_t = u_{xx} + u^p - |u_x|^q\,, & x \in (-1,1),\ t > 0, \\ u(\pm 1, t) = 0, & t > 0, \\ u(x,0) = u_0(x), & x \in (-1,1). \end{cases} \tag{1}$$

Key words and phrases. — Chipot-Weissler equation, blow up, finite difference scheme, numerical blow up set, asymptotic behaviours, numerical convergence.

Here $p>1,\ 1\leq q\leq \dfrac{2p}{p+1}$ and u_0 is a positive function which is compatible with the boundary condition. It is well known that for some initial data, this problem blows up in a finite time. Problem (1) was studied for the first time by Chipot and Weissler in [**1**], since then, the phenomenon of blow up for different problems has been the issue of intensive study, see for example [**3**],[**4**],[**6**],[**7**],[**8**] and the references therein. There exists many theoretical studies on the question of the occurence of blow up, but from a numerical point of view, many interesting numerical questions for problem (1) are not treated.

We define the blow-up set for problem (1) as:

$$B(u)=\{x\in[-1,1];\ \ \exists\ (x_n,t_n)\to(x,T^*)\ \text{ such that } u(x_n,t_n)\to+\infty \text{ as } n\to+\infty\}\,.$$

It is proved in [**2**] that the solution of (1) blows up only at the central point, that is:

$$\exists\ \ T^*<+\infty \text{ such that } \lim_{t\to T^*}u(t,0)=+\infty \text{ but } \lim_{t\to T^*}u(t,x)<\infty \text{ when } x\neq 0.$$

In [**5**], we have conctructed a finite difference scheme whose solution satisfies the same properties as the exact solution and moreover, we have proved that its solution blows up in a finite time. In this paper and for the same scheme, we show the convergence of the numerical solution to the continuous one under some restrictions on p and q, and we study the asymptotic behaviour of the solution near its singularity. We prove that the numerical solution can blow up at more than one point, while a one point blow up is known to occur in the continuous problem. More precisely, we show that even if a difference solution blows up, its values remain bounded up to the moment of blow up except at the maximum point and its adjacent points, moreover, the number of blow up points depends, in a way, on the value of the parameter q.

We recall the scheme studied in [**5**], for $j=1,...,N_n$ and $n\geq 0$ we have

$$\begin{cases} \dfrac{u_j^{n+1}-u_j^n}{\tau_n}=\dfrac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{h_n^2}+(u_j^n)^p-\dfrac{1}{(2h_n)^q}\left|u_{j+1}^n-u_{j-1}^n\right|^{q-1}\left|u_{j+1}^{n+1}-u_{j-1}^{n+1}\right|, \\ u_j^0=u_0(x_j), \\ u_0^n=u_{N_n+1}^n=0. \end{cases} \tag{2}$$

We denote by $U^n:=(u_0^n,...,u_{N_n+1}^n)^t$ the numerical solution of (2), and

$$\|U^n\|_\infty=\max_{1\leq j\leq N_n}|u_j^n|$$

the L^∞ norm of U^n.
Here the notation u_j^n is employed to denote the approximation of $u(x_j,t^n)$ for $x_j\in[-1,1]$ and $t^n\geq 0$. Also, we fix other notations as follow:

1. τ : size parameter for the variable time mesh τ_n.

2. h : size parameter for the variable space mesh h_n.
3. t^n: n-th time step on $t>0$ determined as:

$$\begin{cases} t_0 = 0 \\ t_n = t^{n-1} + \tau_{n-1} = \sum\limits_{k=1}^{n-1} \tau_k; \ n \geq 1. \end{cases}$$

4. x_j: j-th net point on $[-1,1]$ determined as:

$$\begin{cases} x_0 = -1, \\ x_j = x_{j-1} + h_n, \ j \geq 1 \text{ and } \ n \geq 0, \\ x_{N_n+1} = 1. \end{cases}$$

We suppose that a spatial net point x_m coincides with the middle point $x=0$.

5. τ_n : discrete time increment of $n-$th step determined by

$$\tau_n = \tau \min\left(1, \|U^n\|_\infty^{-p+1}\right).$$

6. h_n : discrete space increment of $n-$th step determined by

$$h_n = \min\left(h, \left(2\,\|U^n\|_\infty^{-q+1}\right)^{\frac{1}{2-q}}\right).$$

7. $N_n = \dfrac{1}{h_n} - 1$ the number of subdivisions of the interval $[-1,1]$.
8. $m = \dfrac{N_n+1}{2}$.

As in [**5**], we suppose that the initial data u_0 satisfies the following conditions:
(A1) u_0 is continuous, nonconstant and nonnegative in $[-1,1]$.
(A2) u_0 is spatially symmetric about $x=0$.
(A3) u_0 is strictly monotone increasing in $[-1,0]$.
(A4) $u_0(-1) = u_0(1) = 0$.
(A5) u_0 is large in the sense that $\|u_0\|_\infty >> 1$.

This paper is organized as follows: In section 2, we state and prove the main results, that is, if $p=2$ and $q=1$ then the solution blows up at the maximum point and the points around it, but remains bounded at all of the rest points, while if $p>2$ and $q<\dfrac{2(p-1)}{p}$, then there is only a single point for the solution to blow up. In section 3, we prove the convergence of the numerical solution to the exact one. In section 4, we give an approximation of the blowing-up time. Finally, in section 5, we present some numerical simulations.

2. Main theorems

In this section, we study the asymptotic behaviour of the difference solution near the maximal point x_m.

Theorem 2.1. — *Let U^n be a solution of (2), we suppose that $h < \dfrac{1}{1+\tau}$. For $p = 2$ and $q = 1$, we have*

$$\lim_{n\to+\infty} u^n_{m-1} = \lim_{n\to+\infty} u^n_{m+1} = +\infty.$$

Proof. — For $j = m-1$, the equation of (2) can be rewritten as

$$\begin{aligned} &(1+2\lambda_n)u^{n+1}_{m-1} \\ = \;& \lambda_n(u^{n+1}_{m-2} + u^{n+1}_m) + u^n_{m-1} + \tau_n(u^n_{m-1})^p - \frac{\tau_n}{(2h_n)^q}\left|u^n_m - u^n_{m-2}\right|^{q-1}\left|u^{n+1}_m - u^{n+1}_{m-2}\right|. \end{aligned} \tag{3}$$

Using positivity and monotony we get

$$(1+2\lambda_n)u^{n+1}_{m-1} \geq \lambda_n u^{n+1}_m + u^n_{m-1} - \frac{\tau_n}{(2h_n)^q}(u^n_m - u^n_{m-2})^{q-1}(u^{n+1}_m - u^{n+1}_{m-2}).$$

We use that

$$u^n_m - u^n_{m-2} \leq 2u^n_m \text{ and } u^{n+1}_m - u^{n+1}_{m-2} \leq 2u^{n+1}_m, \tag{4}$$

we obtain

$$u^{n+1}_{m-1} \geq \frac{\lambda_n}{1+2\lambda_n}u^{n+1}_m + \frac{1}{1+2\lambda_n}u^n_{m-1} - \frac{\tau_n}{h_n^q(1+2\lambda_n)}(u^n_m)^{q-1}u^{n+1}_m. \tag{5}$$

Furthermore from (3) for $j = m$, we have

$$u^{n+1}_m = \frac{2\lambda_n}{1+2\lambda_n}u^{n+1}_{m-1} + \frac{u^n_m}{1+2\lambda_n}\left(1 + \tau_n(u^n_m)^{p-1}\right), \tag{6}$$

which implies that

$$u^{n+1}_m \geq \frac{u^n_m}{1+2\lambda_n}. \tag{7}$$

Using (5), (6) and (7) we get for $p = 2$ and $q = 1$

$$u^{n+1}_{m-1} \geq \frac{\lambda_n}{(1+2\lambda_n)^2}u^n_m + \frac{1}{1+2\lambda_n}u^n_{m-1} - \frac{\tau_n}{h_n(1+2\lambda_n)}\left[\frac{2\lambda_n}{1+2\lambda_n}u^{n+1}_{m-1} + \frac{u^n_m}{1+2\lambda_n}(1+\tau_n u^n_m)\right].$$

Then,

$$\left(1 + \frac{2\tau_n\lambda_n}{h_n(1+2\lambda_n)^2}\right)u^{n+1}_{m-1} \geq \frac{\lambda_n}{(1+2\lambda_n)^2}u^n_m + \frac{1}{1+2\lambda_n}u^n_{m-1} - \frac{\tau_n u^n_m(1+\tau_n u^n_m)}{h_n(1+2\lambda_n)^2},$$

which implies that

$$u^{n+1}_{m-1} \geq \frac{\lambda_n h_n u^n_m + h_n(1+2\lambda_n)u^n_{m-1} - \tau_n u^n_m(1+\tau_n u^n_m)}{h_n(1+2\lambda_n)^2 + 2\tau_n\lambda_n}. \tag{8}$$

Since the solution blows up, then we have $u_m^n > 1$, moreover

$$\tau_n = \frac{\tau}{u_m^n} \text{ and } h_n = \min(\sqrt{2}, h) = h.$$

Then

$$\lambda_n = \frac{\tau}{h^2 u_m^n}, \quad h_n\lambda_n = \frac{\tau}{h u_m^n} \text{ and } \tau_n\lambda_n = \frac{\tau^2}{h^2 (u_m^n)^2}.$$

Hence, (8) implies

$$u_{m-1}^{n+1} \geq \frac{\dfrac{\tau}{h} + h\left(1 + \dfrac{2\tau}{h^2 u_m^n}\right) u_{m-1}^n - \tau(1+\tau)}{h\left(1 + \dfrac{2\tau}{h^2 u_m^n}\right)^2 + \dfrac{2\tau^2}{(u_m^n)^2 h^2}}.$$

As we have

$$\lim_{n\to+\infty} u_m^n = +\infty,$$

then we get

$$\lim_{n\to+\infty} u_{m-1}^{n+1} \geq \frac{\dfrac{\tau}{h} + h \lim\limits_{n\to+\infty} u_{m-1}^n - \tau(1+\tau)}{h}.$$

If we assume that $\lim\limits_{n\to+\infty} u_{m-1}^n \neq +\infty$, let $l = \lim\limits_{n\to+\infty} u_{m-1}^n$, then we have

$$l \geq \frac{\tau}{h^2} + l - \frac{\tau(1+\tau)}{h} \Rightarrow \frac{\tau}{h}\left(\frac{1}{h} - (1+\tau)\right) \leq 0,$$

which is a contradiction because $h < \dfrac{1}{1+\tau}$.

Therefore, we have

$$\lim_{n\to+\infty} u_{m-1}^n = +\infty,$$

and using symmetry we get the result of Theorem 2.1. $\square$

The next important result for this paper is mentioned in the next theorem:

Theorem 2.2. — *Let U^n be the solution of (2), we suppose that $p \geq 2$ and $1 \leq q < \dfrac{2(p-1)}{p}$.*

(a) *If $p = 2$ and $q = 1$ then*

$$\lim_{n\to+\infty} u_{m-2}^n < +\infty.$$

(b) *If $p > 2$ and $q < \dfrac{2(p-1)}{p}$ then*

$$\lim_{n\to+\infty} u_{m-1}^n < +\infty.$$

Proof. — Let prove **(a):** In (2), if we take $p=2,\ q=1$ and $j=m-2$, we get

$$\begin{aligned}\frac{u_{m-2}^{n+1}-u_{m-2}^{n}}{\tau_n} &= \frac{u_{m-1}^{n+1}-2u_{m-2}^{n+1}+u_{m-3}^{n+1}}{h_n^2}+\left(u_{m-2}^{n}\right)^2-\frac{1}{2h_n}\left(u_{m-1}^{n+1}-u_{m-3}^{n+1}\right)\\ &\leq \frac{u_{m-1}^{n+1}-2u_{m-2}^{n+1}+u_{m-3}^{n+1}}{h_n^2}+\left(u_{m-2}^{n}\right)^2,\end{aligned}$$

but $u_{m-3}^{n+1}-u_{m-2}^{n+1}<0$, then

$$\frac{u_{m-2}^{n+1}-u_{m-2}^{n}}{\tau_n}\leq \frac{u_{m-1}^{n+1}-u_{m-2}^{n+1}}{h_n^2}+\left(u_{m-2}^{n}\right)^2,$$

which implies that

$$(1+\lambda_n)u_{m-2}^{n+1}\leq \lambda_n u_{m-1}^{n+1}+\left(1+\tau_n u_{m-2}^{n}\right)u_{m-2}^{n}. \tag{9}$$

In the other hand, in (3) if we take $j=m-1$, we get

$$\frac{u_{m-1}^{n+1}-u_{m-1}^{n}}{\tau_n}\leq \frac{u_{m-2}^{n+1}-2u_{m-1}^{n+1}+u_{m}^{n+1}}{h_n^2}+\left(u_{m-1}^{n}\right)^2,$$

but $u_{m-2}^{n+1}-u_{m-1}^{n+1}<0$, then

$$\frac{u_{m-1}^{n+1}-u_{m-1}^{n}}{\tau_n}\leq \frac{-u_{m-1}^{n+1}+u_{m}^{n+1}}{h_n^2}+\left(u_{m-1}^{n}\right)^2,$$

which implies that

$$(1+\lambda_n)u_{m-1}^{n+1}\leq \lambda_n u_{m}^{n+1}+\left(1+\tau_n u_{m-1}^{n}\right)u_{m-1}^{n},$$

and then

$$u_{m-1}^{n+1}\leq \frac{\lambda_n u_{m}^{n+1}+\left(1+\tau_n u_{m-1}^{n}\right)u_{m-1}^{n}}{1+\lambda_n}. \tag{10}$$

Next, if we recall (6) for $p=2$ we get

$$u_{m}^{n+1}=\frac{2\lambda_n}{1+2\lambda_n}u_{m-1}^{n+1}+\frac{1+\tau_n u_m^n}{1+2\lambda_n}u_m^n. \tag{11}$$

Putting (11) in (10) we get

$$u_{m-1}^{n+1}\leq \frac{\lambda_n}{1+\lambda_n}\left[\frac{2\lambda_n}{1+2\lambda_n}u_{m-1}^{n+1}+\frac{1+\tau_n u_m^n}{1+2\lambda_n}u_m^n\right]+\frac{\left(1+\tau_n u_{m-1}^{n}\right)}{1+\lambda_n}u_{m-1}^{n},$$

which implies that

$$\left(1-\frac{2\lambda_n^2}{(1+\lambda_n)(1+2\lambda_n)}\right)u_{m-1}^{n+1}\leq \frac{\lambda_n(1+\tau_n u_m^n)}{(1+2\lambda_n)(1+\lambda_n)}u_m^n+\frac{1+\tau_n u_{m-1}^{n}}{1+\lambda_n}u_{m-1}^{n},$$

and then

$$u_{m-1}^{n+1}\leq \frac{\lambda_n(1+\tau_n u_m^n)u_m^n+(1+2\lambda_n)(1+\tau_n u_{m-1}^{n})u_{m-1}^{n}}{1+3\lambda_n}. \tag{12}$$

Now, putting (12) in (9), we get

$$\begin{aligned} u_{m-2}^{n+1} &\leq \frac{\lambda_n}{1+\lambda_n}\left[\frac{\lambda_n(1+\tau_n u_m^n)u_m^n+(1+2\lambda_n)(1+\tau_n u_{m-1}^n)u_{m-1}^n}{1+3\lambda_n}\right]+\frac{\left(1+\tau_n u_{m-2}^n\right)u_{m-2}^n}{1+\lambda_n} \\ &= \frac{\left(1+\tau_n u_{m-2}^n\right)u_{m-2}^n}{1+\lambda_n}+\frac{\lambda_n^2(1+\tau_n u_m^n)u_m^n+\lambda_n(1+2\lambda_n)(1+\tau_n u_{m-1}^n)u_{m-1}^n}{(1+\lambda_n)(1+3\lambda_n)}. \end{aligned}$$

Then

$$u_{m-2}^{n+1} \leq A_n u_{m-2}^n + B_n, \tag{13}$$

here we have put

$$A_n = \frac{\left(1+\tau_n u_{m-2}^n\right)}{1+\lambda_n}$$

and

$$B_n = \frac{\lambda_n^2(1+\tau_n u_m^n)u_m^n+\lambda_n(1+2\lambda_n)(1+\tau_n u_{m-1}^n)u_{m-1}^n}{(1+\lambda_n)(1+3\lambda_n)}.$$

Then the inequality (13) implies by iterations that

$$\begin{aligned} u_{m-2}^n &\leq A_{n-1}u_{m-2}^{n-1}+B_{n-1} \\ &\leq A_{n-1}A_{n-2}u_{m-2}^{n-2}+A_{n-1}B_{n-2}+B_{n-1} \\ &\quad \vdots \\ &\leq u_{m-2}^0\prod_{k=0}^{n-1}A_k+\sum_{k=0}^{n-2}\left(B_k\prod_{i=k+1}^{n-1}A_i\right)+B_{n-1} \\ &\leq u_{m-2}^0\prod_{k=0}^{n}A_k+\sum_{k=0}^{n-2}B_k\prod_{i=0}^{n-1}A_i+B_{n-1} \\ &\leq u_{m-2}^0\prod_{k=0}^{n}A_k+\sum_{k=0}^{n-2}B_k\prod_{k=0}^{n}A_k+B_{n-1}\left(\prod_{k=0}^{n}A_k\right) \\ &\leq u_{m-2}^0\prod_{k=0}^{n}A_k+\sum_{k=0}^{n-1}B_k\prod_{k=0}^{n}A_k \\ &\leq u_{m-2}^0\prod_{k=0}^{n}A_k+\sum_{k=0}^{n}B_k\prod_{k=0}^{n}A_k \\ &\leq \left(u_{m-2}^0+\sum_{k=0}^{n}B_k\right)\prod_{k=0}^{n}A_k. \end{aligned}$$

To ensure boundedness of u_{m-2}^n we shall prove that

$$\sum_{n\geq 0}B_n<+\infty \text{ and } \prod_{n\geq 0}A_n<+\infty.$$

To do this, we need the next lemma:

Lemma 2.3. — *We define the sequence* $a_n = \dfrac{u^n_{m-1}}{u^n_m}$.

1. *For* $p = 2$ *and* $q = 1$*, we assume that* $\sup_n u^n_{m-1} > \dfrac{3}{h^2}(1+\tau)$*, then* $(a_n)_n$ *converges to 0.*
2. *For* $p > 2$ *and* $q < \dfrac{2(p-1)}{p}$*, we have*
   (a) $(a_n)_n$ *converges to 0.*
   (b) $\lim_{n\to+\infty} \dfrac{a_{n+1}}{a_n} = \dfrac{1}{1+\tau}$.
   (c) $\lim_{n\to+\infty} \dfrac{u^{n+1}_m}{u^n_m} = 1+\tau > 1$.

Proof. — First of all, we look for some useful relations between a_n and a_{n+1}. We recall (3) for $p \geq 2$ and $1 \leq q < \dfrac{2(p-1)}{p} < \dfrac{2p}{p+1}$. We use the same calculations as (12) we obtain that (3) implies

$$u^{n+1}_{m-1} \leq \frac{\lambda_n(1+\tau_n(u^n_m)^{p-1})u^n_m + (1+2\lambda_n)(1+\tau_n(u^n_{m-1})^{p-1})u^n_{m-1}}{1+3\lambda_n}. \tag{14}$$

Using (6), we get

$$\begin{aligned} a_{n+1} &= \frac{u^{n+1}_{m-1}}{u^{n+1}_m} \\ &= \frac{1+2\lambda_n}{\dfrac{2\lambda_n u^{n+1}_{m-1} + (1+\tau_n(u^n_m)^{p-1})u^n_m}{u^{n+1}_{m-1}}} \\ &= \frac{1+2\lambda_n}{2\lambda_n + \dfrac{(1+\tau_n(u^n_m)^{p-1})u^n_m}{u^{n+1}_{m-1}}}. \end{aligned} \tag{15}$$

By substituting (14) into (15) we get:

$$\begin{aligned} a_{n+1} &\leq (1+2\lambda_n)\left\{2\lambda_n + \frac{(1+3\lambda_n)(1+\tau_n(u^n_m)^{p-1})u^n_m}{\lambda_n(1+\tau_n(u^n_m)^{p-1})u^n_m + (1+2\lambda_n)(1+\tau_n(u^n_{m-1})^{p-1})u^n_{m-1}}\right\}^{-1} \\ &= \frac{\lambda_n(1+2\lambda_n)(1+\tau_n(u^n_m)^{p-1})u^n_m + (1+2\lambda_n)^2(1+\tau_n(u^n_{m-1})^{p-1})u^n_{m-1}}{(1+3\lambda_n+2\lambda_n^2)(1+\tau_n(u^n_m)^{p-1})u^n_m + 2\lambda_n(1+2\lambda_n)(1+\tau_n(u^n_{m-1})^{p-1})u^n_{m-1}} \\ &\leq \frac{\lambda_n(1+\tau_n(u^n_m)^{p-1})u^n_m + (1+2\lambda_n)(1+\tau_n(u^n_{m-1})^{p-1})u^n_{m-1}}{(1+\lambda_n)(1+\tau_n(u^n_m)^{p-1})u^n_m + 2\lambda_n(1+\tau_n(u^n_{m-1})^{p-1})u^n_{m-1}} \\ &\leq \frac{\lambda_n(1+\tau_n(u^n_m)^{p-1}) + (1+2\lambda_n)(1+\tau_n(u^n_{m-1})^{p-1})a_n}{(1+\lambda_n)(1+\tau_n(u^n_m)^{p-1}) + 2\lambda_n(1+\tau_n(u^n_{m-1})^{p-1})a_n}. \end{aligned}$$

But we have $\tau_n = \dfrac{\tau}{(u_m^n)^{p-1}}$, then

$$a_{n+1} \leq \frac{\lambda_n(1+\tau) + (1+2\lambda_n)(1+\tau(a_n)^{p-1})a_n}{(1+\lambda_n)(1+\tau) + 2\lambda_n(1+\tau(a_n)^{p-1})a_n}. \tag{16}$$

And finally we get

$$\frac{a_{n+1}}{a_n} \leq \frac{\lambda_n(1+\tau)(a_n)^{-1} + (1+2\lambda_n)(1+\tau(a_n)^{p-1})}{(1+\lambda_n)(1+\tau) + 2\lambda_n(1+\tau(a_n)^{p-1})a_n}. \tag{17}$$

In the other hand, using (3) and (4) we get

$$u_{m-1}^{n+1} \geq \frac{\lambda_n u_m^{n+1} + (1+\tau_n(u_{m-1}^n)^{p-1})u_{m-1}^n}{1+2\lambda_n} - \frac{\tau_n}{h_n^q(1+2\lambda_n)}(u_m^n)^{q-1}u_m^{n+1}.$$

By using (6), we have

$$\begin{aligned} u_{m-1}^{n+1} \geq & \frac{2\lambda_n^2 u_{m-1}^{n+1} + \lambda_n(1+\tau_n(u_m^n)^{p-1})u_m^n + (1+2\lambda_n)(1+\tau_n(u_{m-1}^n)^{p-1})u_{m-1}^n}{(1+2\lambda_n)^2} \\ & - \frac{2\lambda_n\tau_n(u_m^n)^{q-1}u_{m-1}^{n+1}}{h_n^q(1+2\lambda_n)^2} - \frac{\tau_n(1+\tau_n(u_m^n)^{p-1})(u_m^n)^q}{h_n^q(1+2\lambda_n)^2}, \end{aligned}$$

which implies

$$\begin{aligned} & \left(1 - \frac{2\lambda_n^2}{(1+2\lambda_n)^2} + \frac{2\lambda_n\tau_n(u_m^n)^{q-1}}{h_n^q(1+2\lambda_n)^2}\right) u_{m-1}^{n+1} \\ \geq \quad & \frac{\lambda_n(1+\tau_n(u_m^n)^{p-1})u_m^n + (1+2\lambda_n)(1+\tau_n(u_{m-1}^n)^{p-1})u_{m-1}^n}{(1+2\lambda_n)^2} \\ & - \frac{\tau_n(1+\tau_n(u_m^n)^{p-1})(u_m^n)^q}{h_n^q(1+2\lambda_n)^2}, \end{aligned}$$

and then

$$\begin{aligned} u_{m-1}^{n+1} \quad \geq \quad & \frac{\lambda_n h_n^q(1+\tau_n(u_m^n)^{p-1})u_m^n + h_n^q(1+2\lambda_n)(1+\tau_n(u_{m-1}^n)^{p-1})u_{m-1}^n}{h_n^q(1+2\lambda_n)^2 - 2h_n^q\lambda_n^2 + 2\lambda_n\tau_n(u_m^n)^{q-1}} \\ & - \frac{\tau_n(1+\tau_n(u_m^n)^{p-1})(u_m^n)^q}{h_n^q(1+2\lambda_n)^2 - 2h_n^q\lambda_n^2 + 2\lambda_n\tau_n(u_m^n)^{q-1}}. \end{aligned} \tag{18}$$

Using (6) and (18), we get

$$\begin{aligned} & a_{n+1} \\ = \quad & \frac{u_{m-1}^{n+1}}{u_m^{n+1}} \\ = \quad & (1+2\lambda_n)\left\{2\lambda_n + \frac{(1+\tau_n(u_m^n)^{p-1})u_m^n}{u_{m-1}^{n+1}}\right\}^{-1} \\ \geq \quad & \frac{(1+\tau_n(u_m^n)^{p-1})u_m^n\left(\lambda_n h_n^q - \tau_n(u_m^n)^{q-1}\right) + h_n^q(1+2\lambda_n)(1+\tau_n(u_{m-1}^n)^{p-1})u_{m-1}^n}{2\lambda_n h_n^q(1+\tau_n(u_{m-1}^n)^{p-1})u_{m-1}^n + h_n^q(1+2\lambda_n)(1+\tau_n(u_m^n)^{p-1})u_m^n}. \end{aligned}$$

Then we can deduce that

$$\frac{a_{n+1}}{a_n} \geq \frac{(1+\tau_n(u_m^n)^{p-1})(\lambda_n - \tau_n h_n^{-q}(u_m^n)^{q-1})a_n^{-1} + (1+2\lambda_n)(1+\tau_n(u_{m-1}^n)^{p-1})}{(1+\tau_n(u_m^n)^{p-1})(1+2\lambda_n) + 2\lambda_n(1+\tau_n(u_{m-1}^n)^{p-1})a_n}.$$

Finally we get

$$\frac{a_{n+1}}{a_n} \geq \frac{(1+\tau)(\lambda_n - \tau h_n^{-q}(u_m^n)^{q-p})a_n^{-1} + (1+2\lambda_n)(1+\tau(a_n)^{p-1})}{(1+\tau)(1+2\lambda_n) + 2\lambda_n(1+\tau(a_n)^{p-1})a_n}. \tag{19}$$

Next, we prove that the sequence $(a_n)_n$ converges to 0. To prove convergence, we only need to show that $\frac{a_{n+1}}{a_n} < 1$.

But

$$\frac{a_{n+1}}{a_n} \leq \frac{\lambda_n(1+\tau)(a_n)^{-1} + (1+2\lambda_n)(1+\tau(a_n)^{p-1})}{(1+\lambda_n)(1+\tau) + 2\lambda_n(1+\tau(a_n)^{p-1})a_n}.$$

Let

$$A := (1+\tau)\lambda_n + (1+2\lambda_n)(1+\tau(a_n)^{p-1})a_n - (1+\lambda_n)(1+\tau)a_n - 2\lambda_n(1+\tau(a_n)^{p-1})a_n^2.$$

We shall prove that $A < 0$.

(1) First of all, we can see that, for $p = 2$ and $q = 1$

$$\begin{aligned}
A &= (1+\tau)\lambda_n + (1+2\lambda_n)(1+\tau a_n)a_n - (1+\lambda_n)(1+\tau)a_n - 2\lambda_n(1+\tau a_n)a_n^2 \\
&= (1+\tau)\lambda_n + (1+\tau a_n)a_n + 2\lambda_n(1+\tau a_n)a_n - (1+\tau)a_n - \lambda_n(1+\tau)a_n - 2\lambda_n(1+\tau a_n)a_n^2 \\
&= \lambda_n\left(1+\tau+2a_n(1+\tau a_n) - (1+\tau)a_n - 2a_n^2(1+\tau a_n)\right) + \tau a_n^2 - \tau a_n \\
&= \lambda_n\left((1+\tau)(1-a_n) + 2a_n(1+\tau a_n)(1-a_n)\right) + \tau a_n(a_n-1) \\
&= (1-a_n)\lambda_n(1+\tau+2a_n(1+\tau a_n)) + \tau a_n(a_n-1)
\end{aligned}$$

Using

$$\begin{cases} a_n = \dfrac{u_{m-1}^n}{u_m^n}, \\ 0 < a_n < 1, \\ a_n < u_{m-1}^n, \\ \tau = \tau_n u_m^n, \\ h_n = h, \end{cases}$$

we get

$$\begin{aligned}
A &= (1-a_n)\lambda_n(1+\tau+2a_n(1+\tau a_n) + \tau a_n(a_n-1) \\
&< \lambda_n(1-a_n)(1+\tau+2(1+\tau)) + \tau_n u_{m-1}^n(a_n-1) \\
&< (1-a_n)\tau_n(3h^{-2}(1+\tau) - u_{m-1}^n)
\end{aligned}$$

Using the condition: $\sup_n \left\{u_{m-1}^n\right\} > 3h^{-2}(1+\tau)$,we can see that $A < 0$, so that $0 \leq a_{n+1} < a_n < 1$, which implies that $\lim_{n\to+\infty} a_n = a$ exists and satisfies $0 \leq a < 1$.

(2) For $p > 2$ and $q < \dfrac{2p-2}{p}$, we can see that

$$\lambda_n - (1+\lambda_n)a_n < 0,$$

if not, then,

$$\frac{\lambda_n}{a_n} \geq 1+\lambda_n \Rightarrow \frac{\tau 2^{\frac{-2}{2-q}}}{u_{m-1}^n}\left(u_m^n\right)^{\frac{2-2p+pq}{2-q}} \geq 1+\lambda_n,$$

which is a contradiction because of $q < \frac{2p-2}{p}$.
Let now,

$$A_1 = \frac{1+\tau}{1+\tau a_n^{p-1}} > 1$$

and

$$A_2 = \frac{2\lambda_n a_n - (1+2\lambda_n)}{\lambda_n - (1+\lambda_n)a_n} < 1.$$

Then it is clear that

$$\begin{aligned}
&A_1 > a_n A_2.\\
&\Rightarrow \frac{1+\tau}{1+\tau a_n^{p-1}} > \frac{a_n(2\lambda_n a_n - (1+2\lambda_n))}{\lambda_n - (1+\lambda_n)a_n}.\\
&\Rightarrow (1+\tau)(\lambda_n - (1+\lambda_n)a_n) < a_n(1+\tau a_n^{p-1})(2\lambda_n a_n - (1+2\lambda_n)).\\
&\Rightarrow (1+\tau)\lambda_n + (1+2\lambda_n)a_n(1+\tau a_n^{p-1}) - (1+\tau)(1+\lambda_n a_n) - 2\lambda_n a_n^2(1+\tau a_n^{p-1}) < 0.\\
&\Rightarrow A < 0.
\end{aligned}$$

So $0 \leq a_{n+1} < a_n < 1$.
We shall prove now that $a = 0$ for all $p > 1$ and $1 \leq q < \dfrac{2(p-1)}{p}$. By reduction to absurdity we suppose that $0 < a < 1$. Letting $n \to \infty$ in (16) we obtain

$$a \leq \frac{1+\tau a^{p-1}}{1+\tau} a < a$$

which is a contradiction. This proves that $a = 0$.
Next we prove that $\lim_{n\to+\infty} \dfrac{a_{n+1}}{a_n} = \dfrac{1}{1+\tau}$, for $p > 2$ and $q < \dfrac{2(p-1)}{p}$
By means of (17) we get

$$\frac{a_{n+1}}{a_n} \leq \frac{\lambda_n(1+\tau)(a_n)^{-1} + (1+2\lambda_n)(1+\tau(a_n)^{p-1})}{(1+\lambda_n)(1+\tau) + 2\lambda_n(1+\tau(a_n)^{p-1})a_n}, \tag{20}$$

but

$$\lambda_n(1+\tau)(a_n)^{-1} = c_1(u_{m-1}^n)^{-1}(u_m^n)^{\frac{-2p+pq+2}{2-q}},$$

where $c_1 = \dfrac{\tau(1+\tau)}{4^{\frac{1}{2-q}}}$.
And for $q < \dfrac{2p-2}{p}$ we have $\dfrac{-2p+pq+2}{2-q} < 0$, then we obtain

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} \leq \frac{1}{1+\tau}. \tag{21}$$

In the other hand, using (19) we get

$$\frac{a_{n+1}}{a_n} \geq \frac{(1+\tau)(\lambda_n - \tau h_n^{-q}(u_m^n)^{q-p})a_n^{-1} + (1+2\lambda_n)(1+\tau(a_n)^{p-1})}{(1+\tau)(1+2\lambda_n) + 2\lambda_n(1+\tau(a_n)^{p-1})a_n},$$

but

$$(\lambda_n - \tau h_n^{-q}(u_m^n)^{q-p})a_n^{-1} = (u_{m-1}^n)^{-1}\left(c_1\,(u_m^n)^{-p+2+\frac{2q-2}{2-q}} - c_2\,(u_m^n)^{q-p+1+\frac{-q(-q+1)}{2-q}}\right),$$

where $c_1, c_2 \in \mathbb{R}$.
And for $q < \dfrac{2(p-1)}{p}$ we have

$$\left(\lambda_n - \tau h_n^{-q}(u_m^n)^{q-p}\right) a_n^{-1} \to 0 \text{ as } n \to +\infty,$$

then we obtain

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} \geq \frac{1}{1+\tau}. \tag{22}$$

Finally from (21) and (22) we deduce that

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \frac{1}{1+\tau} < 1.$$

To finish the proof of Lemma 2.3 we shall prove that for all $p > 2$ and $q < \dfrac{2(p-1)}{p}$ we have $\lim\limits_{n\to+\infty} \dfrac{u_m^{n+1}}{u_m^n} = 1+\tau$.
From (6), we know that

$$(1+2\lambda_n)u_m^{n+1} - 2\lambda_n u_{m-1}^{n+1} = (1+\tau_n(u_m^n)^{p-1})u_m^n,$$

which implies

$$1 + 2\lambda_n - 2\lambda_n \frac{u_{m-1}^{n+1}}{u_m^{n+1}} = (1+\tau_n(u_m^n)^{p-1})\frac{u_m^n}{u_m^{n+1}}.$$

Then

$$1 + 2\lambda_n - 2\lambda_n a_{n+1} = (1+\tau)\frac{u_m^n}{u_m^{n+1}}.$$

So

$$1 = (1+\tau)\lim_{n\to+\infty} \frac{u_m^n}{u_m^{n+1}}.$$

This implies $\lim\limits_{n\to+\infty} \dfrac{u_m^{n+1}}{u_m^n} = 1+\tau > 1$. This achieve the proof of Lemma 2.3. $\square$

Now we can finish the proof of Theorem 2.2. We have showed that

$$u_{m-2}^n \le \left(u_{m-2}^0 + \sum_{k=0}^{n} B_k \right) \prod_{k=0}^{n} A_k,$$

where

$$A_n = \frac{\left(1 + \tau_n u_{m-2}^n\right)}{1 + \lambda_n}$$

and

$$B_n = \frac{\lambda_n^2 (1 + \tau_n u_m^n) u_m^n + \lambda_n (1 + 2\lambda_n)(1 + \tau_n u_{m-1}^n) u_{m-1}^n}{(1 + \lambda_n)(1 + 3\lambda_n)}.$$

Using that $u_m^n >> 1$, we can see that for $p = 2$ and $q = 1$ we have

$$\tau_n = \frac{\tau}{u_m^n} \text{ and } h_n = h.$$

Then

$$A_n \le 1 + \tau_n u_{m-2}^n = 1 + \tau \frac{u_{m-2}^n}{u_m^n} \le 1 + \tau \frac{u_{m-1}^n}{u_m^n} = 1 + \tau a_n,$$

and

$$\begin{aligned} B_n &= \frac{\lambda_n^2 (1 + \tau_n u_m^n) u_m^n + \lambda_n (1 + 2\lambda_n)(1 + \tau_n u_{m-1}^n) u_{m-1}^n}{(1 + \lambda_n)(1 + 3\lambda_n)} \\ &\le \lambda_n^2 (1 + \tau_n u_m^n) u_m^n + \lambda_n (1 + 2\lambda_n)(1 + \tau_n u_{m-1}^n) u_{m-1}^n \\ &\le \lambda_n^2 (1 + \tau) u_m^n + \lambda_n (1 + 2\lambda_n)(1 + \tau_n u_m^n) u_{m-1}^n \\ &= \frac{\tau^2}{h^4} (1 + \tau) \frac{1}{u_m^n} + \frac{\tau}{h^2 u_m^n} (1 + 2 \frac{\tau}{h^2 u_m^n})(1 + \tau) u_{m-1}^n \\ &\le c^2 (1 + \tau) \frac{u_{m-1}^n}{u_m^n} + c(1 + 2c)(1 + \tau) \frac{u_{m-1}^n}{u_m^n} \\ &\le c(1 + \tau) \left(c + (1 + 2c)\right) \frac{u_{m-1}^n}{u_m^n} \\ &\le c(1 + \tau)(1 + 3c) \frac{u_{m-1}^n}{u_m^n}, \end{aligned}$$

with $c = \dfrac{\tau}{h^2}$.
But we have

$$\lim_{n \to +\infty} \frac{a_{n+1}}{a_n} < 1 \text{ and } a_n > 0,$$

then

$$0 < \sum_{n \ge 0} a_n < +\infty.$$

In the other hand, for all $c > 0$, we have $\sum_{n\geq 0} ca_n < +\infty$, then

$$1 < \prod_{n\geq 0}(1 + ca_n) < +\infty.$$

We deduce from this that

$$0 < \sum_{n\geq 0} B_n \leq c(1+\tau)(1+3c)\sum_{n\geq 0} a_n < +\infty,$$

and

$$1 < \prod_{n\geq 0} A_n \leq \prod_{n\geq 0}(1 + \tau a_n) < +\infty,$$

which implies that

$$\lim_{n\to+\infty} u^n_{m-2} < +\infty.$$

Now we will prove the second result of Theorem 2.2, that is:

$$\text{If } p > 2 \text{ and } q < \frac{2(p-1)}{p} \text{ then } \lim_{n\to+\infty} u^n_{m-1} < +\infty.$$

In (2), we put $j = m - 1$ and we consider the quantity

$$\begin{aligned} u^{n+1}_{m-1} - u^n_{m-1} &\leq \lambda_n(u^{n+1}_{m-2} - 2u^{n+1}_{m-1} + u^{n+1}_m) + \tau_n(u^n_{m-1})^p \\ &= G_n + H_n, \end{aligned}$$

where

$$\begin{aligned} G_n &= \lambda_n(u^{n+1}_{m-2} - 2u^{n+1}_{m-1} + u^{n+1}_m) \\ &= c(u^n_m)^{-p+1+\frac{2(q-1)}{2-q}}(u^{n+1}_{m-2} - 2u^{n+1}_{m-1} + u^{n+1}_m) \\ &= c(u^n_m)^{-p+1+\frac{2(q-1)}{2-q}} u^{n+1}_m\left(\frac{u^{n+1}_{m-2}}{u^{n+1}_m} - 2\frac{u^{n+1}_{m-1}}{u^{n+1}_m} + 1\right) \\ &= c(u^n_m)^{-p+2+\frac{2(q-1)}{2-q}} \frac{u^{n+1}_m}{u^n_m}\left(\frac{u^{n+1}_{m-2}}{u^{n+1}_{m-1}} a_{n+1} - 2a_{n+1} + 1\right) \\ &= c(u^n_m)^{-p+2+\frac{2q-2}{2-q}} \frac{u^{n+1}_m}{u^n_m}\left(1 - a_{n+1}\left(2 - \frac{u^{n+1}_{m-2}}{u^{n+1}_{m-1}}\right)\right) \\ &> 0, \end{aligned}$$

with $c := \dfrac{\tau}{2^{\frac{2}{2-q}}}$ and

$$H_n = \tau_n(u^n_{m-1})^p = \tau u^n_m (a_n)^p > 0.$$

Therefore, using Lemma 2.3, we get

$$\lim_{n\to+\infty} \frac{G_{n+1}}{G_n} = (1+\tau)^{-p+2+\frac{2q-2}{q-2}} < 1 \text{ for } q < \frac{2(p-1)}{p},$$

which implies
$$\sum_{n\geq 0} G_n < +\infty.$$
Also
$$\lim_{n\to+\infty} \frac{H_{n+1}}{H_n} = (1+\tau)^{-p+1} < 1 \text{ for } p > 2,$$
which implies
$$\sum_{n\geq 0} H_n < +\infty.$$
Hence we get the boundedness of u_{m-1}^n from:
$$\begin{aligned} 0 < u_{m-1}^n &= \sum_{k=1}^{n} (u_{m-1}^k - u_{m-1}^{k-1}) + u_{m-1}^0 \\ &\leq \sum_{k=1}^{n} (G_{k-1} + H_{k-1}) + u_{m-1}^0 \\ &\leq \sum_{k=0}^{+\infty} (G_k + H_k) + u_{m-1}^0 \\ &< +\infty. \end{aligned}$$
Thus we have completed the proof of Theorem 2.2. □

3. Convergence

In this section we prove the convergence of the numerical solution given by (2), to the nodal values of the solution u of (1) on each fixed interval time $[0, T], T < T^*$ as far as the smoothness of u is guaranteed.

Lemma 3.1. — *Let u be the classical solution of (1) and U^n be the numerical solution of (2). Let T be an arbitrary number such that $0 < T < T^*$. Then there exist positive constants C_0, C_1, depending only on T and u_0, such that*
(A) *For $p > 2$ and $q < \dfrac{2(p-1)}{p}$*
$$\max_{1\leq j\leq m-2} \left|u_j^n - u(x_j, t^n)\right| \leq C_0 h^{3-q}$$
holds so far as $t_n < T$.
(B) *For $p > 1$ and $q = 1$*
$$\max_{1\leq j\leq m-1} \left|u_j^n - u(x_j, t^n)\right| \leq C_1 h^2$$
holds so far as $t_n < T$.

Before studying local convergence, we prove the consistency of the scheme.

3.1. Consistency. — For all $1 \leq j \leq N_n$, we define

$$\begin{aligned}\epsilon_j^n &= \frac{u(x_j,t^{n+1})-u(x_j,t^n)}{\tau_n} - \frac{u(x_{j+1},t^{n+1})-2u(x_j,t^{n+1})+u(x_{j-1},t^{n+1})}{h_n^2} \\ &- (u(x_j,t^n))^p + \left|\frac{u(x_{j+1},t^n)-u(x_{j-1},t^n)}{2h_n}\right|^{q-1}\left|\frac{u(x_{j+1},t^{n+1})-u(x_{j-1},t^{n+1}}{2h_n}\right|.\end{aligned}$$

We use Taylor formula, we obtain

$$\frac{\partial u}{\partial t}(x_j,t^n) = \frac{u(x_j,t^{n+1})-u(x_j,t^n)}{\tau_n} - \frac{\tau_n}{2}\frac{\partial^2 u}{\partial t^2}(x_j,t^n+\tau_n\theta_1). \tag{23}$$

$$\begin{aligned}\frac{\partial u}{\partial x}(x_j,t^n) &= \frac{u(x_{j+1},t^n)-u(x_{j-1},t^n)}{2h_n} - \frac{h_n^2}{3}\frac{\partial^3 u}{\partial x^3}(x_j+h_n\theta_2,t^n) \\ &\quad -\frac{h_n^2}{3}\frac{\partial^3 u}{\partial x^3}(x_j-h_n\theta_3,t^n).\end{aligned} \tag{24}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}(x_j,t^n) &= \frac{u(x_{j+1},t^n)-2u(x_j,t^n)+u(x_{j-1},t^n)}{h_n^2} - \frac{h_n^2}{24}\frac{\partial^4 u}{\partial x^4}(x_j+h_n\theta_4,t^n) \\ &\quad -\frac{h_n^2}{24}\frac{\partial^4 u}{\partial x^4}(x_j-h_n\theta_5,t^n).\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}(x_j,t^n) &= \frac{\partial^2 u}{\partial x^2}(x_j,t^{n+1}) - \tau_n\frac{\partial^3 u}{\partial t\partial x^2}(x_j,t^n+\tau_n\theta_6) \\ &= \frac{u(x_{j+1},t^{n+1})-2u(x_j,t^{n+1})+u(x_{j-1},t^{n+1})}{h_n^2} + \frac{h_n^2}{24}\frac{\partial^4 u}{\partial x^4}(x_j+h_n\theta_7,t^{n+1}) \\ &\quad +\frac{h_n^2}{24}\frac{\partial^4 u}{\partial x^4}(x_j-h_n\theta_8,t^{n+1}) + \tau_n\frac{\partial^3 u}{\partial t\partial x^2}(x_j,t^n+\tau_n\theta_6).\end{aligned} \tag{25}$$

where $0 < \theta_i < 1$ for $i = 1, ..., 8$.
We define

$$F = \left|\frac{\partial u}{\partial x}(x_j,t^n)\right|^q - \left|\frac{u(x_{j+1},t^n)-u(x_{j-1},t^n)}{2h_n}\right|^q.$$

We use the mean value theorem, the monotony and the symmetry of the exact solution proved in Theorem 2.3 and Theorem 2.4 in [**5**], then there exists A between $\frac{\partial u}{\partial x}(x_j,t^n)$ and $\frac{u(x_{j+1},t^n)-u(x_{j-1},t^n)}{2h_n}$ such that

$$\begin{aligned}&\left|\left|\frac{\partial u}{\partial x}(x_j,t^n)\right|^q - \left|\frac{u(x_{j+1},t^n)-u(x_{j-1},t^n)}{2h_n}\right|^q\right| \\ =\ &\left|q\,|A|^{q-1}\left(\frac{\partial u}{\partial x}(x_j,t^n) - \frac{u(x_{j+1},t^n)-u(x_{j-1},t^n)}{2h_n}\right)\right| \\ =\ &q\,|A|^{q-1}\,o(h_n^2),\end{aligned}$$

with

$$\left|\frac{\partial u}{\partial x}(x_j,t^n)-A\right|\le\left|\frac{\partial u}{\partial x}(x_j,t^n)-\frac{u(x_{j+1},t^n)-u(x_{j-1},t^n)}{2h_n}\right|\le o(h_n^2).$$

Since $\left|\frac{\partial u}{\partial x}\right|$ is bounded before blow up by [**1**], then we can deduce that A is bounded too. So we can write that

$$\begin{aligned}\left|\frac{\partial u}{\partial x}(x_j,t^n)\right|^q &= \left|\frac{u(x_{j+1},t^n)-u(x_{j-1},t^n)}{2h_n}\right|^q+o(h_n^2) \qquad (26)\\ &= \left|\frac{u(x_{j+1},t^n)-u(x_{j-1},t^n)}{2h_n}\right|^{q-1}\left|\frac{u(x_{j+1},t^n)-u(x_{j-1},t^n)}{2h_n}\right|+o(h_n^2)\\ &= \left|\frac{u(x_{j+1},t^n)-u(x_{j-1},t^n)}{2h_n}\right|^{q-1}\left|\frac{\partial u}{\partial x}(x_j,t^n)+o(h_n^2)\right|+o(h_n^2)\\ &= \left|\frac{u(x_{j+1},t^n)-u(x_{j-1},t^n)}{2h_n}\right|^{q-1}\left|\frac{\partial u}{\partial x}(x_j,t^n)\right|+o(h_n^2)\\ &= \left|\frac{u(x_{j+1},t^n)-u(x_{j-1},t^n)}{2h_n}\right|^{q-1}\left|\frac{\partial u}{\partial x}(x_j,t^{n+1})\right|+o(\tau_n)+o(h_n^2).\end{aligned}$$

Then

$$\begin{aligned}\left|\frac{\partial u}{\partial x}(x_j,t^n)\right|^q &= \left|\frac{u(x_{j+1},t^n)-u(x_{j-1},t^n)}{2h_n}\right|^{q-1}\left|\frac{u(x_{j+1},t^{n+1})-u(x_{j-1},t^{n+1})}{2h_n}\right|\\ &\quad +o(\tau_n)+o(h_n^2). \qquad (27)\end{aligned}$$

We replace (23), (25) and (27) in ϵ_j^n we obtain

$$\begin{aligned}\epsilon_j^n &= \frac{\partial u}{\partial t}(x_j,t^n)+\frac{\tau_n}{2}\frac{\partial^2 u}{\partial t^2}(x_j,t^n+\tau_n\theta_1)-\frac{\partial^2 u}{\partial x^2}(x_j,t^n)-\tau_n\frac{\partial^3 u}{\partial t\partial x^2}(x_j,t^n+\tau_n\theta_4)\\ &\quad -\frac{h_n^2}{24}\frac{\partial^4 u}{\partial x^4}(x_j+h_n\theta_5,t^{n+1})-\frac{h_n^2}{24}\frac{\partial^4 u}{\partial x^4}(x_j-h_n\theta_6,t^{n+1})-(u(x_j,t^n))^p\\ &\quad +\left|\frac{\partial u}{\partial x}(x_j,t^n)\right|^q+o(\tau_n)+o(h_n^2).\end{aligned}$$

If we put

$$R_1=\max_{x,t}\left|\frac{1}{2}\frac{\partial^2 u}{\partial t^2}(x,t)+\frac{\partial^3 u}{\partial t\partial x^2}(x,t)\right| \text{ and } R_2=\frac{1}{12}\max_{x,t}\left|\frac{\partial^4 u}{\partial x^4}(x,t)\right|,$$

we can deduce that

$$\max_{1\le j\le N_n}\epsilon_j^n\le C_1\tau_n+C_2h_n^2,$$

with $C_1\tau_n=R_1\tau_n+o(\tau_n)$ and $C_2h_n^2=R_2h_n^2+o(h_n^2)$.

3.2. Local convergence. — Let $e_j^n = u_j^n - u(x_j, t^n)$ for $j = 1, ..., m-2$.
(A): Using (23), (25) and 26 we get

$$\begin{aligned}
&\frac{u(x_j,t^{n+1}) - u(x_j,t^n)}{\tau_n} - \frac{u(x_{j+1},t^{n+1}) - 2u(x_j,t^{n+1}) + u(x_{j-1},t^{n+1})}{h_n^2} - (u(x_j,t^n))^p \\
&\quad + \left|\frac{u(x_{j+1},t^n) - u(x_{j-1},t^n)}{2h_n}\right|^q \\
&= \frac{\tau_n}{2}\frac{\partial^2 u}{\partial t^2}(x_j, t^n + \theta_1\tau_n) - \tau_n\frac{\partial^3 u}{\partial t \partial x^2}(x_j, t^n + \theta_6\tau_n) \\
&\quad -\frac{h_n^2}{24}\left\{\frac{\partial^4 u}{\partial x^4}(x_j + \theta_7 h_n, t^{n+1}) + \frac{\partial^4 u}{\partial x^4}(x_j - \theta_8 h_n, t^{n+1})\right\} + o(h_n^2).
\end{aligned}$$

Let

$$\begin{aligned}
r_j^n &:= -\frac{\tau_n}{2}\frac{\partial^2 u}{\partial t^2}(x_j, t^n + \theta_1\tau_n) + \tau_n\frac{\partial^3 u}{\partial t \partial x^2}(x_j, t^n + \theta_6\tau_n) \\
&\quad +\frac{h_n^2}{24}\left\{\frac{\partial^4 u}{\partial x^4}(x_j + \theta_7 h_n, t^{n+1}) + \frac{\partial^4 u}{\partial x^4}(x_j - \theta_8 h_n, t^{n+1})\right\} + o(h_n^2)
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{u(x_j,t^{n+1}) - u(x_j,t^n)}{\tau_n} - \frac{u(x_{j+1},t^{n+1}) - 2u(x_j,t^{n+1}) + u(x_{j-1},t^{n+1})}{h_n^2} - (u(x_j,t^n))^p \\
&\quad + \left|\frac{u(x_{j+1},t^n) - u(x_{j-1},t^n)}{2h_n}\right|^q = -r_j^n. \qquad (28)
\end{aligned}$$

Using (2), we have

$$\frac{u_j^{n+1} - u_j^n}{\tau_n} - \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h_n^2} - (u_j^n)^p + \frac{1}{(2h_n)^q}|u_{j+1}^n - u_{j-1}^n|^{q-1}|u_{j+1}^{n+1} - u_{j-1}^{n+1}| = 0. \quad (29)$$

From (28) and (29), e_j^n satisfies

$$\begin{aligned}
&\frac{e_j^{n+1} - e_j^n}{\tau_n} - \frac{e_{j+1}^{n+1} - 2e_j^{n+1} + e_{j-1}^{n+1}}{h_n^2} - \big((u_j^n)^p - u(x_j,t^n)^p\big) \\
&\quad + \frac{1}{(2h_n)^q}|u_{j+1}^n - u_{j-1}^n|^{q-1}|u_{j+1}^{n+1} - u_{j-1}^{n+1}| - \left|\frac{u(x_{j+1},t^n) - u(x_{j-1},t^n)}{2h_n}\right|^q \\
&= r_j^n.
\end{aligned}$$

By the mean-value Theorem, for $f(X) = X^p$, we get

$$\begin{aligned}
(u_j^n)^p - (u(x_j,t^n))^p &= f(u_j^n) - f(u(x_j,t^n)) \\
&= f'(u(x_j,t^n) + \theta_9 e_j^n)e_j^n,
\end{aligned}$$

for some $\theta_9 \in [0,1]$. Then we obtain

$$\begin{aligned} & \frac{e_j^{n+1}-e_j^n}{\tau_n} - \frac{e_{j+1}^{n+1}-2e_j^{n+1}+e_{j-1}^{n+1}}{h_n^2} \\ = \; & f'(u(x_j,t^n)+\theta_9 e_j^n)e_j^n - \frac{1}{(2h_n)^q}|u_{j+1}^n - u_{j-1}^n|^{q-1}|u_{j+1}^{n+1}-u_{j-1}^{n+1}| \\ & + \left|\frac{u(x_{j+1},t^n)-u(x_{j-1},t^n)}{2h_n}\right|^q + r_j^n. \end{aligned}$$

Using (26) we get

$$\begin{aligned} & \frac{e_j^{n+1}-e_j^n}{\tau_n} - \frac{e_{j+1}^{n+1}-2e_j^{n+1}+e_{j-1}^{n+1}}{h_n^2} \\ = \; & f'(u(x_j,t^n)+\theta_5 e_j^n)e_j^n - \frac{1}{(2h_n)^q}|u_{j+1}^n - u_{j-1}^n|^{q-1}|u_{j+1}^{n+1}-u_{j-1}^{n+1}| + \left|\frac{\partial u}{\partial x}(x_j,t^n)\right|^q + r_{1j}^n. \end{aligned}$$

with $r_{1j}^n = r_j^n + o(h_n^2)$.
Let:

$$E^n = \max_{1\le j\le m-2}\left|e_j^n\right|, \qquad U = \max_{x,t}|u(x,t)|, \qquad V = \max_{x,t}\left|\frac{\partial u}{\partial x}(x,t)\right|,$$

$$W = \frac{2}{3}\max_{x,t}\left|\frac{\partial^3 u}{\partial x^3}(x,t)\right|, \qquad K = f'(U+1),$$

and

$$R = \frac{\lambda_n}{2}\max_{x,t}\left|\frac{\partial^2 u}{\partial t^2}(x,t)\right| + \lambda_n \max_{x,t}\left|\frac{\partial^2 u}{\partial x^2}(x,t)\right| + \frac{1}{12}\max_{x,t}\left|\frac{\partial^4 u}{\partial x^4}(x,t)\right| + o(1) + o(\lambda_n).$$

But from (24) we have

$$\begin{aligned} & \left|\frac{\partial u}{\partial x}(x,t) - \frac{u_{j+1}^n - u_{j-1}^n}{2h_n}\right| \\ = \; & \left|\frac{u(x_{j+1},t^n)-u(x_{j-1},t^n)}{2h_n} - \frac{h_n^2}{3}\frac{\partial^3 u}{\partial x^3}(x_j+\theta h_n,t^n) + \frac{h_n^2}{3}\frac{\partial^3 u}{\partial x^3}(x_j-\theta h_n,t^n) - \frac{u_{j+1}^n - u_{j-1}^n}{2h_n}\right| \\ = \; & \left|\frac{e_{j-1}^n - e_{j+1}^n}{2h_n} - \frac{h_n^2}{3}\frac{\partial^3 u}{\partial x^3}(x_j+\theta h_n,t^n) - \frac{h_n^2}{3}\frac{\partial^3 u}{\partial x^3}(x_j-\theta h_n,t^n)\right| \\ \le \; & \frac{E^n}{h_n} + h_n^2 W. \end{aligned} \tag{30}$$

Then by (30) and the mean value theorem, for $g(X) = |X|^q$, we get

$$\begin{aligned} & \left|\left|\frac{\partial u}{\partial x}(x_j,t^n)\right|^q - \left|\frac{u_{j+1}^n - u_{j-1}^n}{2h_n}\right|^q\right| \\ \le \; & qg'\left(\frac{\partial u}{\partial x}(x_j,t^n) + \theta\left(\frac{E^n}{h_n} + h_n^2 W\right)\right)\left(\frac{E^n}{h_n} + h_n^2 W\right). \end{aligned} \tag{31}$$

In the other hand, for $1 \le j \le m-2$ we have,

$$
\begin{aligned}
&\left|\left|\frac{u_{j+1}^n - u_{j-1}^n}{2h_n}\right|^q - \left|\frac{u_{j+1}^n - u_{j-1}^n}{2h_n}\right|^{q-1}\left|\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2h_n}\right|\right| \\
= &\left|\frac{u_{j+1}^n - u_{j-1}^n}{2h_n}\right|^{q-1}\left(\frac{u_{j+1}^n - u_{j-1}^n}{2h_n} - \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2h_n}\right) \\
\le &\left|\frac{u_{j+1}^n - u_{j-1}^n}{2h_n}\right|^{q-1}\left(\frac{e_{j+1}^n - e_{j-1}^n}{2h_n} - \frac{e_{j+1}^{n+1} - e_{j-1}^{n+1}}{2h_n} + o(\tau_n) + o(h_n^2)\right) \\
\le &\left|\frac{u_{j+1}^n - u_{j-1}^n}{2h_n}\right|^{q-1}\left(\frac{e_{j+1}^n - e_{j-1}^n}{2h_n}\right) + \left|\frac{u_{j+1}^n - u_{j-1}^n}{2h_n}\right|^{q-1}\left(\frac{e_{j+1}^{n+1} - e_{j-1}^{n+1}}{2h_n}\right) \\
&\quad + \big(o(\tau_n) + o(h_n^2)\big)\left|\frac{u_{j+1}^n - u_{j-1}^n}{2h_n}\right|^{q-1}. \qquad (32)
\end{aligned}
$$

Then from (32) and (31) we get

$$
\begin{aligned}
&\frac{e_j^{n+1} - e_j^n}{\tau_n} - \frac{e_{j+1}^{n+1} - 2e_j^{n+1} + e_{j-1}^{n+1}}{h_n^2} \\
\le\ & f'(u(x_j, t^n) + \theta_9 e_j^n)e_j^n + r_{1j}^n + \big(o(\tau_n) + o(h_n^2)\big)\left|\frac{u_{j+1}^n - u_{j-1}^n}{2h_n}\right|^{q-1} \\
+\ & qg'\left(\frac{\partial u}{\partial x}(x_j, t^n) + \theta\left(\frac{E^n}{h_n} + h_n^2 W\right)\right)\left(\frac{E^n}{h_n} + h_n^2 W\right) + \left|\frac{u_{j+1}^n - u_{j-1}^n}{2h_n}\right|^{q-1}\left(\frac{e_{j+1}^n - e_{j-1}^n}{2h_n}\right) \\
&+\left|\frac{u_{j+1}^n - u_{j-1}^n}{2h_n}\right|^{q-1}\left(\frac{e_{j+1}^{n+1} - e_{j-1}^{n+1}}{2h_n}\right).
\end{aligned}
$$

Let $M := \|U^n\|_\infty = u_m^n$. Finally we obtain

$$
\begin{aligned}
\frac{E^{n+1} - E^n}{\tau_n} \le\ & KE^n + h_n^2 R + \big(o(\tau_n) + o(h_n^2)\big)\left(\frac{M}{h_n}\right)^{q-1} \\
+\ & qg'\left(V + \theta\left(\frac{E^n}{h_n} + h_n^2 W\right)\right)\left(\frac{E^n}{h_n} + h_n^2 W\right) \\
=\ & E^n\left(K + \frac{q}{h_n}g'\left(V + \theta\left(\frac{E^n}{h_n} + h_n^2 W\right)\right)\right) \\
+\ & h_n^2\left(R + (o(\lambda_n) + o(1))\left(\frac{M}{h_n}\right)^{q-1} + Wqg'\left(V + \theta\left(\frac{E^n}{h_n} + h_n^2 W\right)\right)\right) \\
=\ & \frac{E^n}{h_n^q}\left(h_n^q K + qg'\left(h_n V + \theta\left(E^n + h_n^3 W\right)\right)\right) \\
+\ & h_n^{3-q}\left(h_n^{q-1}R + (o(\lambda_n) + o(1))\,M^{q-1} + Wqg'\left(h_n V + \theta\left(E^n + h_n^3 W\right)\right)\right).
\end{aligned}
$$

Let

$$\begin{aligned} B &= h_n^q K + qg'\left(h_n V + \theta\left(E^n + h_n^3 W\right)\right) \\ C &= h_n^{q-1} R + \left(o(\lambda_n) + o(1)\right) M^{q-1} + Wqg'\left(h_n V + \theta\left(E^n + h_n^3 W\right)\right). \end{aligned}$$

Then,

$$\begin{aligned} E^{n+1} &\leq \left(1 + \tau_n \frac{B}{h_n^q}\right) E^n + \tau_n h_n^{3-q} C \\ &\leq \left(1 + \tau_n NB\right) E^n + \tau_n h_n^{3-q} C \\ &\leq \exp(NBT) h_n^{3-q} CT. \end{aligned}$$

With N is constant such that

$$\text{For } \ t_n < T \text{ and } h_n = \left(2M^{-q+1}\right)^{\frac{1}{2-q}} \text{ we have: } \quad \frac{1}{h_n^q} = \frac{M^{\frac{q(q-1)}{2-q}}}{2^{\frac{q}{2-q}}} := N,$$

which is bounded by Theorem 2.2. Then we get

$$\max_{1\leq j\leq m-2} \left|u_j^n - u(x_j, t^n)\right| \leq C_0(T) h^{3-q}.$$

Now, we will prove the last part of the lemma.
(B): We do the same thing for $p > 1$ and $q = 1$, we get for $j = 1, ..., m-1$

$$\begin{aligned} &\frac{e_j^{n+1} - e_j^n}{\tau_n} - \frac{e_{j+1}^{n+1} - 2e_j^{n+1} + e_{j-1}^{n+1}}{h_n^2} \\ = \ & f'(u(x_j, t^n) + \theta_9 e_j^n) e_j^n - \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2h_n} + \frac{u(x_{j+1}, t^n) - u(x_{j-1}, t^n)}{2h_n} + r_j^n \\ = \ & f'(u(x_j, t^n) + \theta_9 e_j^n) e_j^n + \frac{e_{j-1}^{n+1} - e_{j+1}^{n+1}}{2h_n} + r_j^n. \end{aligned}$$

And then

$$\begin{aligned} &\frac{E^{n+1} - E^n}{\tau_n} \leq KE^n + h_n^2 R. \\ \Rightarrow \quad & E^{n+1} \leq \tau_n KE^n + \tau_n h_n^2 R. \\ \Rightarrow \quad & E^{n+1} \leq \exp(KT) h_n^2 RT. \end{aligned}$$

And finally we obtain

$$\max_{1\leq j\leq m-1} \left|u_j^n - u(x_j, t^n)\right| \leq C_1(T) h^2.$$

4. Approximation of the blowing up time

In this section, we give an idea about the numerical blow-up time. First of all we recall a result of Souplet and Weissler [**9**]

Theorem 4.1. — *Let $\psi \in W_0^{1,s}(\Omega)$, (s large enough), with $\psi \geq 0$ and $\psi \neq 0$.*

1. *There exists some $\lambda_0 = \lambda_0(\psi) > 0$ such that for all $\lambda > \lambda_0$, the solution of (1) with initial data $\phi = \lambda\psi$ blows up in finite time in $W^{1,s}$ norm.*
2. *There is some $C > 0$ such that*
$$T^*(\lambda\psi) \leq \frac{C}{(\lambda\,|\psi|_\infty)^{p-1}}, \quad \lambda \to \infty.$$
3. $$T^*(\lambda\psi) \geq \frac{1}{(p-1)(\lambda\,|\psi|_\infty)^{p-1}}.$$

We define now
$$T^*_{num} := \sum_{n\geq 0} \tau_n \tag{33}$$
and call it the numerical blow-up time. In [**5**], we have proved that
$$u_m^n \geq \left(\frac{1+\tau}{1+\tau 2^{\frac{-q}{2-q}}\,(u_m^0)^{\frac{-2p+q(1+p)}{2-q}}}\right)^n u_m^0.$$
which implies that
$$\frac{1}{(u_m^n)^{p-1}} \leq \frac{1}{\left(\frac{1+\tau}{1+\tau 2^{\frac{-q}{2-q}}(u_m^0)^{\frac{-2p+q(1+p)}{2-q}}}\right)^{n(p-1)}}(u_m^0)^{-p+1}. \tag{34}$$
Using (33) and (34) we get
$$\begin{aligned}
T^*_{num} &= \tau\sum_{n\geq 0}\frac{1}{(u_m^n)^{p-1}} \\
&\leq \frac{\tau}{(u_m^0)^{p-1}}\sum_{n\geq 0}\left(\frac{1}{\left(\frac{1+\tau}{1+\tau 2^{\frac{-q}{2-q}}(u_m^0)^{\frac{-2p+q(1+p)}{2-q}}}\right)^{p-1}}\right)^n \\
&= \frac{\tau}{(u_m^0)^{p-1}}\sum_{n\geq 0}\left(\left(\frac{1+\tau 2^{\frac{-q}{2-q}}\,(u_m^0)^{\frac{-2p+q(1+p)}{2-q}}}{1+\tau}\right)^{p-1}\right)^n \\
&= \frac{\tau}{(u_m^0)^{p-1}}\frac{1}{1-\left(\frac{1+\tau 2^{\frac{-q}{2-q}}\,(u_m^0)^{\frac{-2p+q(1+p)}{2-q}}}{1+\tau}\right)^{p-1}} := T^{**}
\end{aligned} \tag{35}$$

5. Numerical simulations

In this section, we present some numerical simulations that illustrate our results. In figure 1, we take $p = 4 > 2$ and $q = 1.3 < \frac{2(p-1)}{p}$, one can see that the solution is bounded in x_{m-1}. Then we take $p = 2$ and $q = 1$, it is clear from figure 2 that the solution blows up in x_{m-1}, and from figure 3, we can see that the solution is bounded in x_{m-2}. Concerning the approximation of the blowing up time, if we take the initial data $u_0(x) = \lambda \sin(\frac{\pi}{2}(x+1))$, with $\lambda > 0$ then $\|u_0\|_\infty = \lambda$. Theoretically we know that

$$T^* \geq \frac{1}{(p-1)\,\|u_0\|_\infty^{p-1}}.$$

Let $g(\lambda) = \dfrac{1}{(p-1)\lambda^{p-1}}$ and $p = 3$. In the next table, and for some values of λ we can see that $T^*_{num} \geq g(\lambda)$ which is compatible with the theoretical result, this is illustrated in figure 4. Also, using (35) and for $\lambda = 10^3$ we have

$$T^*_{num} \approx 5.067.10^{-7} \leq T^{**} = 5.075.10^{-7}.$$

λ	10	10^2	10^3	10^4	10^5
$g(\lambda)$	5.10^{-3}	5.10^{-5}	5.10^{-7}	5.10^{-9}	5.10^{-11}
T^*_{num}	$5.177.10^{-3}$	$5.068.10^{-5}$	$5.067.10^{-7}$	$5.075.10^{-9}$	$5.058.10^{-11}$

TABLE 1. Comparison of the function $g(\lambda)$ with the numerical blow up time T^*_{num}.

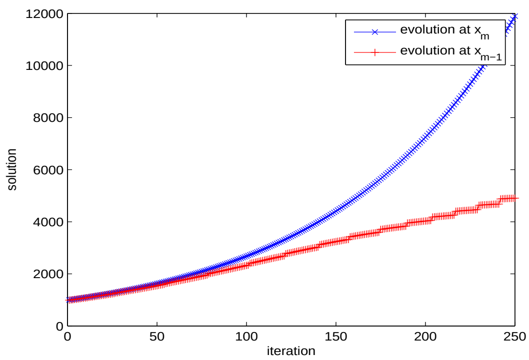

FIGURE 1. Evolution of the numerical solution at x_m and x_{m-1} for $p = 4$ and $q = 1.3$

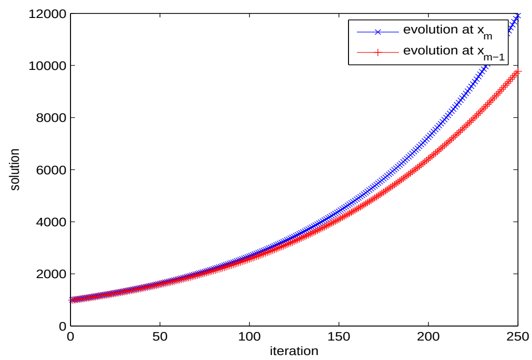

FIGURE 2. Evolution of the numerical solution at x_m and x_{m-1} for $p = 2$ and $q = 1$.

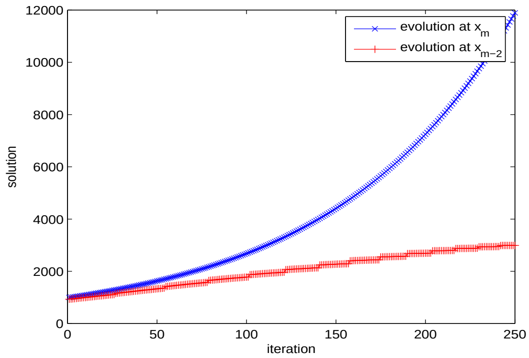

FIGURE 3. Evolution of the numerical solution at x_m and x_{m-2} for $p = 2$ and $q = 1$.

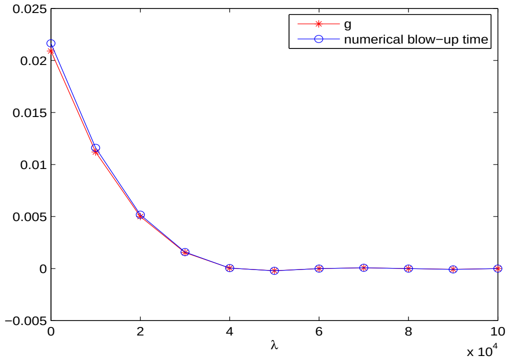

FIGURE 4. Graphics of $g(\lambda)$ and approximation of the numerical blow-up time for $p = 3$.

6. Conclusion

We have showed that when $p = 2$ and $q = 1$, the finite difference solution blows up at more than one point and that when $p > 2$ and $q < \dfrac{2(p-1)}{p}$, the only numerical blow up point is the mid-point $x = 0$. This is an interesting phenomena in view of the fact that the solution of the corresponding PDE blows up only at one point $x = 0$ for any $p > 1$ and $1 \le q \le \dfrac{2p}{p+1}$. Remark that for $1 < p < 2$ and $\dfrac{2(p-1)}{p} \le q < \dfrac{2p}{p+1}$, we have no idea about the boundedness of u^n_{m-1} and u^n_{m-2}.

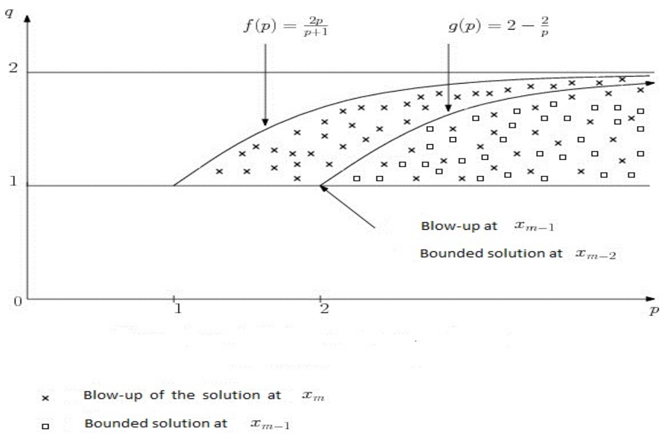

FIGURE 5. Graphics of the asymptotic behaviours of the solution near the blowing-up point.

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